The GL_1-case of my conjecture i.e. geometric class field theory for the curve

<u>Classical case</u>: k alg. closed field. X/k smooth projective curve $\pi_1(X)^{ab} = \pi_1(\underbrace{Jac(X)}_{Pic^\circ(X)})$

<u>Ex</u>: $k=\mathbb{C}$. This is equivalent to Abel's theorem.

$$Pic^\circ(X)(\mathbb{C}) \xrightarrow{\sim} H^0(X,\Omega^1_X)^* / H_1(X,\mathbb{Z})$$

$$[P_1]-[P_2] \longmapsto \left[\left(\omega \longmapsto \int_{P_2}^{P_1} \omega\right) \bmod H_1(X,\mathbb{Z})\right]$$

In fact $Jac(X)$ abelian variety $\Rightarrow Jac(X)^{an} = T_0\, Jac(X) / \pi_1(Jac(X))$

$H^1(X,\mathcal{O}_X) = H^0(X,\Omega^1_X)^*$ ↑ Serre duality

via the theorem

$\pi_1(X)^{ab} = H_1(X,\mathbb{Z})$

Geometric Langlands point of view:

Dualize $\to$ look at characters $\pi_1 \to \overline{\mathbb{Q}_\ell}^\times$

$\mathcal{E}$ = rk. 1 $\overline{\mathbb{Q}_\ell}$-étale local system on X

$$\Big\downarrow$$

$\underbrace{\mathcal{F}}_{}$ = rk. 1 $\overline{\mathbb{Q}_\ell}$ - local system on $\mathrm{Jac}(X)$

automorphic sheaf associated to $\mathcal{E}$

$* d \geq 1$ $\quad \mathrm{Div}^d_X = X^d / \mathfrak{S}_d$ = Hilbert scheme of degree d effective divisors.

Cartier

$\mathrm{Div}^1_X = X$

For $d \geq 2$: $\pi_1(\mathrm{Div}^d_X) = \pi_1(X)^{ab}$ (general fact, valid in topology for a finite CW complexe and assoc.)

$$\Sigma^d : X^d \xrightarrow{\text{finite flat}} \mathrm{Div}^d_X$$

$$(x_1, \dots, x_d) \longmapsto \sum_i [x_i]$$

$$\mathcal{E}^{(d)} := \Big(\underbrace{\Sigma^d_* \, \mathcal{E}^{\boxtimes d}}_{\text{sheaf + action of } \mathfrak{S}_d}\Big)^{\mathfrak{S}_d} = \text{rk. } 1 \ \overline{\mathbb{Q}}_\ell\text{-local system on } \mathrm{Div}^d_X$$

(perverse sheaf in general if $\mathcal{E}$ is not rk. 1)

$$AJ^d : \mathrm{Div}^d_X \longrightarrow \mathrm{Pic}^d_X$$
$$D \longmapsto \mathcal{O}(D)$$

For $d >> 0$ ($d > 2g-2$ so that $H^1(X, \mathcal{L}) = 0$ if $\deg \mathcal{L} = d$)

AJ^d is an étale locally trivial fibration in $\mathbb{P}^{d-g}_{k}$ simply connected.

$\Rightarrow$ for $d>>0$ $\mathcal{E}^{(d)}$ descends along AJ^d to a rk. 1 $\overline{\mathbb{Q}}_\ell$-local system $\mathcal{F}^{(d)}$ on Pic^d i.e. $\mathcal{E}^{(d)} \overset{!}{=} (AJ^d)^* \mathcal{F}^{(d)}$

One checks $\coprod_{d > 2g-2} \mathcal{F}^{(d)}$ is compatible with the monoid structure on $\coprod_{d > 2g-2} \mathrm{Pic}^d$

i.e. if $m : \mathrm{Pic}^{>2g-2} \times \mathrm{Pic}^{>2g-2} \to \mathrm{Pic}^{>2g-2}$

$$m^*\left(\coprod_{d>2g-2} \mathcal{F}^{(d)}\right) \xrightarrow{\sim} \coprod_{d>2g-2} \mathcal{F}^{(d)} \boxtimes \coprod_{d>2g-2} \mathcal{F}^{(d)}$$

(+ cocycle relations.....)

$\mathrm{Pic}^{>2g-2}$ generates $\mathrm{Pic} \Rightarrow \mathcal{F}$ extends naturally to a rk. 1 $\overline{\mathbb{Q}}_\ell$ - local system on $\mathrm{Pic} = \coprod_{d \in \mathbb{Z}} \mathrm{Pic}^d$

Compatible with the group law.

Take $\mathcal{F} := \mathcal{F}^{(0)}$.

$$\begin{array}{ccc} \mathcal{E} & \rightsquigarrow & \mathcal{E}^{(d)} \text{ for } d \gg 0 \\ \downarrow & & \big\downarrow \text{ descent via } AJ^d \\ \mathcal{F} = \mathcal{F}^{(0)} & \leftsquigarrow & \mathcal{F}^{(d)} \end{array}$$

go back in degree 0 via the group law of Pic_X.

Rem: One can replace $\underbrace{\mathrm{Pic}_X}_{\text{Picard scheme}}$ by $\underbrace{\mathcal{P}ic_X}_{\text{Picard stack}}$

in all of this since $\mathcal{P}ic_X \xrightarrow[\mathbb{G}_m\text{-gerb}]{} \underbrace{\mathrm{Pic}_X}_{\text{coarse moduli space}}$

$+ \ \mathbb{G}_m$ connected $\Rightarrow$ $\overline{\mathbb{Q}}_\ell$-loc. sys. $/\mathrm{Pic}_X \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$-loc. sys. $/\mathcal{P}ic_X$

(same π_1).

this will be false in our context: we will have to use $\mathcal{P}ic$.

The relative curve

$$E \begin{cases} \to \mathbb{F}_q((\pi)) \\ \to [E:\mathbb{Q}_p] < +\infty, \quad \mathcal{O}_E/\pi = \mathbb{F}_q \end{cases}$$

$\mathrm{Perf}_{\mathbb{F}_q} \ni S \longmapsto X_S$ adic space

$\underbrace{\mathrm{Perf}_{\mathbb{F}_q}}_{\mathbb{F}_q\text{-perfectoid spaces}}$

$$X_S \longrightarrow \mathrm{Spa}\, E$$

"in some sense" X_S = family of curves $\Big(X_{\underbrace{k(s),\, k(s)^+}_{\text{adic curve defined before when } k(s)^+ = k(s)^\circ}}\Big)_{s \in S}$

perfectoid field

$*$ $E = \mathbb{F}_q((\pi))$ - Easy:

$$Y_S = \mathbb{D}^*_S = \{0 < |\pi| < 1\} \subset \mathbb{A}^1_S$$

$\varphi = \mathrm{Frob}_S$

$$Y_S \longrightarrow \mathbb{D}^*_{\mathbb{F}_q} = \mathrm{Spa}\left(\mathbb{F}_q((\pi))\right)$$

$$\boxed{X_S = \mathbb{D}^*_S / \varphi^{\mathbb{Z}}}$$

$*$ $E | \mathbb{Q}_p$:

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Suppose $S = \mathrm{Spa}(R, R^+)$ affinoid perfectoid

$$W_{O_E}(\underbrace{R^+}_{\text{perfect}}) = \left\{ \sum_{n\geq 0} [x_n]\pi^n \,/\, x_n \in R^+ \right\}$$

φ given by $\varphi\left(\sum_n [x_n]\pi^n\right) = \sum_n [x_n^q]\pi^n$

Let $\varpi \in R^{\circ\circ} \cap R^\times$ be a pseudo-uniformizing element.

$$Y_{R,R^+} := \mathrm{Spa}\left(W_{O_E}(R^+), W_{O_E}(R^+)\right) \setminus V(\pi[\varpi])$$

$$\mathbb{D}^*_{R,R^+} = \mathrm{Spa}\left(R^+[[\pi]], R^+[[\pi]]\right) \setminus V(\pi.\varpi)$$

Fix a power multiplicative norm $\|\cdot\| : R \longrightarrow \mathbb{R}_+$ defining the topology.

invert

$\mathcal{O}(Y_{R,R^+})$ = Frechet Completion of

$$W_{O_E}(R^+)\left[\frac{1}{\pi}, \frac{1}{[\varpi]}\right] = \left\{ \sum_{n>>-\infty} [\underset{\substack{\cap \\ R}}{x_n}]\pi^n \,/\, \sup_n \|x_n\| < +\infty \right\}$$

w.r.t. Gauss norms

$$\left\| \sum_n [x_n] \pi^n \right\|_\rho = \sup_n \|x_n\| \rho^n \quad , \rho \in]0,1[$$

$$W_{\mathcal{O}_E}(R^+) = \mathcal{O}\left(Y_{R,R^+}\right)^+$$ ← functions bounded by 1.

$$X_{R,R^+} = Y_{R,R^+} / \varphi^{\mathbb{Z}}$$

Rem: The fact that this is an adic space (i.e. Huber's presheaf is a sheaf needs a proof, this is not evident).

Can glue this construction to $S \longmapsto X_S$ functorial.

The Picard stack

$\mathrm{Perf}_{\mathbb{F}_q}$ equipped with the pro-étale topology.

<u>Recall:</u> * Affinoid perfectoid spaces have filtered projective limits.

$$\varprojlim_i \mathrm{Spa}(R_i, R_i^+) = \mathrm{Spa}(R_\infty, R_\infty^+)$$

where $R_\infty^+ = \widehat{\varinjlim_i R_i^+}$

ϖ-adic completion where ϖ is any pseudo unif. element of R_i for some, sent to $\varinjlim_i R_i^+$.

$$R_\infty = R_\infty^+\left[\tfrac{1}{\varpi}\right]$$

* A morphism of perfectoid spaces $T \longrightarrow S$ is pro-étale if locally on T and S it is of the form

$$\begin{array}{c} \varprojlim_{i \geq i_0} \mathrm{Spa}(R_i, R_i^+) \\ \downarrow \\ \mathrm{Spa}(R_{i_0}, R_{i_0}^+) \end{array}$$

with étale transition morphisms in the projective system

* A morphism $T \xrightarrow{f} S'$ pro-étale is a covering if $\forall U \subset S'$ quasi-compact open $\exists V \subset T$ quasi-compact open such that $f(V) \supset U$.

$\rightsquigarrow$ pro-étale topology (Big site)

Ex: $s \in S'$ $\quad \varprojlim_{U \ni s} U = \mathrm{Spa}\left(k(s), k(s)^+\right) \hookrightarrow S'$ pro-étale

$\rightarrow$ analogy for schemes $\varprojlim_{U \ni s} U = \mathrm{Spec}\left(\mathcal{O}_{S,s}\right) \hookrightarrow S$ pro-étale

Def: Pic = Picard stack on $\mathrm{Perf}_{\mathbb{F}_q}$ + pro-étale top.

s.t. $\forall S' \in \mathrm{Perf}_{\mathbb{F}_q}$ $\quad \mathrm{Pic}(S)$ = groupoid of line bundles on $X_{S'}$.

Recall: A = locally profinite set

$\rightsquigarrow \underline{A}$ = pro-étale sheaf on $\mathrm{Perf}_{\mathbb{F}_q}$ defined by

$$\underline{A}(S) = \mathcal{C}^0(|S|, A)$$

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$\forall S' \in \mathrm{Perf}_{\mathbb{F}_q}$ $\quad \underline{A}_{S'}$ is representable by an

$$\begin{array}{c} \underline{A}_{S'} \\ \downarrow \\ S' \end{array}$$

S-perfectoid space.

$$S = \mathrm{Spa}(R, R^+) \Rightarrow \underline{A}_S = \mathrm{Spa}\big(\underbrace{\mathcal{C}^0(A, R), \mathcal{C}^0(A, R^+)}_{\text{perfectoid algebra.}}\big)$$

Th (Kedlaya–Liu): $S' \in \mathrm{Perf}_{\mathbb{F}_q}$

Fiberwise on S' degree 0 line bundles on $X_{S'}$ $\xrightarrow{\sim}$ rk. 1 $\underline{E}$-local systems on S'

global sections functor

$\mathcal{L}/X_{S'}$ s.t. $\forall s \in S'$ $\mathcal{L}|X_{k(s), k(s)^+}$ is degree 0

locally constant pro-étale sheaves of $\underline{E}$-modules of rk. 1

$$\mathcal{L}/X_{S'} \longmapsto \left[T/S \mapsto H^0(X_T, \mathcal{L}) \right]$$

$\to$ $\mathcal{L}/X_S$ fiberwise on S' of degree $0 \Rightarrow \exists\, \tilde{S} \to S$ pro-étale covering s.t. $\mathcal{L}_{|X_{\tilde{S}}} \simeq \mathcal{O}_{X_{\tilde{S}}}$ trivial line bundle

$+ \; H^0(X_S, \mathcal{O}_{X_S}) = \underline{E}(S')$.

<u>Generalization</u> (à la Narasimhan–Seshadri):

$$\left\{ \begin{array}{l} \text{Fiberwise on } S' \text{ slope } 0 \\ \text{semi-stable vector bundles} \\ \text{on } X_S \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{l} \text{pro-étale } \underline{E}\text{-local} \\ \text{systems of finite rk} \end{array} \right\}$$

global sections functor.

Corollary:

$$\mathrm{Pic}^{\circ} \xrightarrow{\sim} \underbrace{[\,\cdot\,/\underline{E}^{\times}]}_{\text{classifying stacks of pro-étale } \underline{E}^{\times}\text{-torsors.}}$$

$$\Rightarrow \quad \mathrm{Pic} \simeq \coprod_{d \in \mathbb{Z}} [\,\cdot\,/\underline{E}^{\times}]$$

twisting by $\mathcal{O}(d)$

$$\underset{\mathrm{Pic}^{d}(S')}{\mathcal{L} \in} \longmapsto \underbrace{\mathrm{Isom}\left(\mathcal{O}(d), \mathcal{L}\right)}_{\text{pro-étale } \underline{E}^{\times}\text{-torsors on } S'.}$$